Tutorial Sheet-4: Counting, Generating Functions and Recurrence Relations

(1) How many ways are there to assign three jobs to five employees if each employee can be given more than one job?

(2) How many ways are there to select three unordered elements from a set with five elements when repetition is allowed?

(3) How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 17,$$

where x_1, x_2, x_3, and x_4 are non-negative integers?

(4) How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 21,$$

where $x_i,\ i = 1, 2, 3, 4, 5$, is a non-negative integer such that

(a) $x_1 \geq 1$?

(b) $x_i \geq 2$ for $i = 1, 2, 3, 4, 5$?

(c) $0 \leq x_1 \leq 10$?

(d) $0 \leq x_1 \leq 3,\ 1 \leq x_2 < 4$, and $x_3 \geq 15$?

(5) How many solutions are there to the inequality

$$x_1 + x_2 + x_3 \leq 11,$$

where x_1, x_2, and x_3 are non-negative integers?[Hint: Introduce an auxiliary variable x_4 such that $x_1 + x_2 + x_3 + x_4 = 11$.]

(6) Show that if there are 30 students in a class, then at least two have last names that begin with the same letter.

(8) How many numbers must be selected from the set $\{1, 3, 5, 7, 9, 11, 13, 15\}$ to guarantee that at least one pair of these numbers add up to 16?

(9) In how many different orders can five runners finish a race if no ties are allowed?

(10) In how many ways can a set of five letters be selected from the English alphabet?

(11) How many subsets with more than two elements does a set with 100 elements have?

(12) (a) What is the generating function for $\{a_k\}$, where a_k is the number of solutions of

$$x_1 + x_2 + x_3 + x_4 = k,$$

when x_1, x_2, x_3, and x_4 are integers with $x_1 \geq 3,\ 1 \leq x_2 \leq 5,\ 0 \leq x_3 \leq 4$, and $x_4 \geq 1$?

(b) Use your answer to part (a) to find a_7?

(13) Use generating functions to solve the recurrence relation $a_k = 7a_{k-1}$ with the initial condition $a_0 = 5$.

(14) Use generating functions to solve the recurrence relation $a_k = 3a_{k-1} + 2$ with the initial condition $a_0 = 1$.

(15) Solve these recurrence relations together with the initial conditions given

(a) $a_n = a_{n-1} + 6a_{n-2}$ for $n \geq 2,\ a_0 = 3,\ a_1 = 6$

(b) $a_n = 7a_{n-1} - 10a_{n-2}$ for $n \geq 2,\ a_0 = 2,\ a_1 = 1$

(c) $a_n = -6a_{n-1} - 9a_{n-2}$ for $n \geq 2,\ a_0 = 3,\ a_1 = -3$

(d) $a_{n+2} = -4a_{n+1} + 5a_n$ for $n \geq 0,\ a_0 = 2,\ a_1 = 8$.